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Explicit formulas for 2nd-order driving terms due to sextupoles and chromatic effects of quadrupoles

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Optimization of nonlinear driving terms have become a useful tool for designing storage rings, especially modern light sources where the strong nonlinearity is dominated by the large chromatic effects of quadrupoles and strong sextupoles for chromaticity control. The Lie algebraic method is well known for computing such driving terms. However, it appears that there was a lack of explicit formulas in the public domain for such computation, resulting in uncertainty and/or inconsistency in widely used codes. This note presents explicit formulas for driving terms due to sextupoles and chromatic effects of quadrupoles, which can be considered as thin elements. The computation is accurate to the 4th-order Hamiltonian and 2nd-order in terms of magnet parameters. The results given here are the same as the APS internal note AOP-TN-2009-020. This internal note has been revised and published here as a Light Source Note in order to get this information into the public domain, since both ELEGANT and OPA are using these formulas.

I. INTRODUCTION

It is well known that the effective Hamiltonian of a storage ring can be concatenated from individual element maps via similarity transformations and BCH formula. Let h_n and f_n^i be the n-th order generator for the ring Hamiltonian and i-th element in the ring, respectively. Then to the 4th Hamiltonian order (i.e., the octupole order),

$$e^{:h_2:}e^{:h_3:}e^{:h_4:}\cdots = \prod_{i=1}(e^{:f_2^i:}e^{:f_3^i:}e^{:f_4^i:}\cdots) \equiv (e^{:f_2^1:}e^{:f_3^1:}e^{:f_4^1:}\cdots)(e^{:f_2^2:}e^{:f_3^2:}e^{:f_4^2:}\cdots)\cdots, \tag{1}$$

and it is straightforward to show that (see [1, 2], for example)

$$\begin{aligned} \mathcal{R} &= e^{:h_2:} = \prod_{i=1}^N e^{:f_2^i:} \equiv e^{:f_2^1:}e^{:f_2^2:}e^{:f_2^3:}\cdots e^{:f_2^N:} \\ h_3 &= \sum_{i=1}^N \tilde{f}_3^i, \\ h_4 &= \sum_{i=1}^N \tilde{f}_4^i + \frac{1}{2}\sum_{j>i=1}^N [\tilde{f}_3^i, \tilde{f}_3^j] \\ &\cdots, \end{aligned} \tag{2}$$

where $\tilde{f}^i$ means $\tilde{f}^i(X) = f^i(R_{i\to N}^{-1}X)$ with $R_{i\to N}^{-1} = R_{N\to i} \leftrightarrow e^{-:f_2^N:}e^{-:f_2^{N-1}:}\cdots e^{-:f_2^{i+1}:}$, the linear transformation matrix from the last element N to the i-th element. $[f, g]$ represents the Poisson bracket of f and g. Note that since $[h_3, h_4]$ is 5th order, $e^{:h_3:}e^{:h_4:} = e^{:h_3+h_4:}$ up to the 4th order.

To reduce the nonlinearity of the ring, one would like to minimize the nonlinear generators h_3 and h_4 as much as possible. Because the linear system is periodic, it is more useful to examine the nonlinear generators using the resonance basis, i.e., the eigenmodes of the linear map. The connection between the normal coordinates (x, p_x, y, p_y), action-angle variables $(J_x, \phi_x, J_y, \phi_y)$, and the resonance basis $(\hat{x}_+, \hat{x}_-, \hat{y}_+, \hat{y}_-)$ are commonly defined as

$$\hat{x}_\pm \equiv \sqrt{2J_x}e^{\pm i\phi_x} = x \mp ip_x, \quad \hat{y}_\pm \equiv \sqrt{2J_y}e^{\pm i\phi_y} = y \mp ip_y. \tag{3}$$

Using the resonance basis, the effective Hamiltonian can be expanded as

$$h_n = \sum_{i_1,i_2,i_3,i_4,i_5\geq 0}^{i_1+i_2+i_3+i_4+i_5=n} h_{i_1i_2i_3i_4i_5}\, \hat{x}_+^{i_1}\, \hat{x}_-^{i_2}\, \hat{y}_+^{i_3}\, \hat{y}_-^{i_4}\, \delta^{i_5}. \tag{4}$$

This note gives the explicit formulas for these coefficients for a simplified storage ring model, where the nonlinear contributions are due to chromatic effects of quadrupoles and sextupoles that can be treated as thin elements. The computation is straightforward but rather tedious, and thus was done with the help of Mathematica^{TM}.

Note that the resonance associated with each term is:

$$(i_1 - i_2)\nu_x + (i_3 - i_4)\nu_y = integer, \tag{5}$$

where $\nu_{x,y}$ are the horizontal and vertical tunes.

II. FORMULAS FOR THE DRIVING TERMS

Here we use the same notations as in [3] and consider thin sextupoles with vector potential

$$V_i = -\frac{1}{3} b_{3i}(x^3 - 3xy^2), \tag{6}$$

and chromatic contribution from quadrupoles

$$V_i = \frac{1}{2} b_{2i}(x^2 - y^2)\delta. \tag{7}$$

The purpose here is to give explicit 2nd-order driving terms, i.e., the coefficients of $h^{(2)}$ in Eq. (93) of [3]. We obtained the same 1st-order expressions as in [3] and explicit 2nd-order expressions as:

$$h_{22000} = \sum \frac{i}{64} \bar{b}_{3i}\bar{b}_{3j}\beta_{xi}^{3/2}\beta_{xj}^{3/2}\left[e^{i\,3(\psi_{xi}-\psi_{xj})} + 3e^{i(\psi_{xi}-\psi_{xj})}\right], \tag{8}$$

$$h_{31000} = h^*_{13000} = \sum \frac{i}{32} \bar{b}_{3i}\bar{b}_{3j}\beta_{xi}^{3/2}\beta_{xj}^{3/2}e^{i\,(3\psi_{xi}-\psi_{xj})}, \tag{9}$$

$$h_{21010} = h_{12100} = 0, \tag{10}$$

$$h_{21100} = h^*_{12010} = 0, \tag{11}$$

$$h_{11110} = \sum \frac{i}{16} \bar{b}_{3i}\bar{b}_{3j}\sqrt{\beta_{xi}\beta_{xj}}\beta_{yi}\left\{\beta_{xj}\left[e^{-i(\psi_{xi}-\psi_{xj})} - e^{i(\psi_{xi}-\psi_{xj})}\right]\right.$$
$$\left. +\beta_{yj}\left[e^{i(\psi_{xi}-\psi_{xj}+2\psi_{yi}-2\psi_{yj})} + e^{-i(\psi_{xi}-\psi_{xj}-2\psi_{yi}+2\psi_{yj})}\right]\right\}, \tag{12}$$

$$h_{11200} = h^*_{11020} = \sum \frac{i}{32} \bar{b}_{3i}\bar{b}_{3j}\sqrt{\beta_{xi}\beta_{xj}}\beta_{yi}\left\{\beta_{xj}\left[e^{-i(\psi_{xi}-\psi_{xj}-2\psi_{yi})} - e^{i(\psi_{xi}-\psi_{xj}+2\psi_{yi})}\right]\right.$$
$$\left. +2\beta_{yj}\left[e^{i(\psi_{xi}-\psi_{xj}+2\psi_{yi})} + e^{-i(\psi_{xi}-\psi_{xj}-2\psi_{yi})}\right]\right\}, \tag{13}$$

$$h_{40000} = h^*_{04000} = \sum \frac{i}{64} \bar{b}_{3i}\bar{b}_{3j}\beta_{xi}^{3/2}\beta_{xj}^{3/2}e^{i(3\psi_{xi}+\psi_{xj})}, \tag{14}$$

$$h_{30010} = h^*_{03100} = 0, \tag{15}$$

$$h_{30100} = h^*_{03010} = 0, \tag{16}$$

$$h_{20020} = h^*_{02200} = \sum \frac{i}{64} \bar{b}_{3i}\bar{b}_{3j}\sqrt{\beta_{xi}\beta_{xj}}\beta_{yi}\left[\beta_{xj}e^{-i(\psi_{xi}-3\psi_{xj}+2\psi_{yi})} - (\beta_{xj} + 4\beta_{yj})e^{i(\psi_{xi}+\psi_{xj}-2\psi_{yi})}\right], \tag{17}$$

$$h_{20110} = h^*_{02110} = \sum \frac{i}{32} \bar{b}_{3i}\bar{b}_{3j}\sqrt{\beta_{xi}\beta_{xj}}\beta_{yi}\left\{\beta_{xj}\left[e^{-i(\psi_{xi}-3\psi_{xj})} - e^{i(\psi_{xi}+\psi_{xj})}\right] + 2\beta_{yj}e^{i(\psi_{xi}+\psi_{xj}+2\psi_{yi}-2\psi_{yj})}\right\}, \tag{18}$$

$$h_{20200} = h^*_{02020} = \sum \frac{i}{64} \bar{b}_{3i}\bar{b}_{3j}\sqrt{\beta_{xi}\beta_{xj}}\beta_{yi}\left[\beta_{xj}e^{-i(\psi_{xi}-3\psi_{xj}-2\psi_{yi})} - (\beta_{xj} - 4\beta_{yj})e^{i(\psi_{xi}+\psi_{xj}+2\psi_{yi})}\right], \tag{19}$$

$$h_{10030} = h^*_{01300} = 0, \tag{20}$$

$$h_{10120} = h^*_{01210} = 0, \tag{21}$$

$$h_{10210} = h^*_{01120} = 0, \tag{22}$$

$$h_{10300} = h^*_{01030} = 0, \tag{23}$$

$$h_{00220} = \sum \frac{i}{64} \bar{b}_{3i}\bar{b}_{3j}\sqrt{\beta_{xi}\beta_{xj}}\beta_{yi}\beta_{yj}\left[e^{i(\psi_{xi}-\psi_{xj}+2\psi_{yi}-2\psi_{yj})} + 4e^{i(\psi_{xi}-\psi_{xj})} - e^{-i(\psi_{xi}-\psi_{xj}-2\psi_{yi}+2\psi_{yj})}\right], \tag{24}$$

$$h_{00310} = h^*_{00130} = \sum \frac{i}{32}\bar{b}_{3i}\bar{b}_{3j}\sqrt{\beta_{xi}\beta_{xj}}\beta_{yi}\beta_{yj}\left[e^{i(\psi_{xi}-\psi_{xj}+2\psi_{yi})} - e^{-i(\psi_{xi}-\psi_{xj}-2\psi_{yi})}\right], \tag{25}$$

$$h_{00400} = h^*_{00040} = \sum \frac{i}{64}\bar{b}_{3i}\bar{b}_{3j}\sqrt{\beta_{xi}\beta_{xj}}\beta_{yi}\beta_{yj}e^{i(\psi_{xi}-\psi_{xj}+2\psi_{yi}+2\psi_{yj})}, \tag{26}$$

$$\begin{aligned} h_{21001} = h^*_{12001} = \sum \Bigg\{ & -\frac{i}{32}\bar{b}_{3i}\bar{b}_{2j}\beta_{xi}^{3/2}\beta_{xj}\left[e^{i\psi_{xi}} + e^{i(3\psi_{xi}-2\psi_{xj})} - 2e^{-i(\psi_{xi}-2\psi_{xj})}\right] \\ & -\frac{i}{16}\bar{b}_{3i}\bar{b}_{3j}\beta_{xi}\beta_{xj}^{3/2}\eta_{xi}\left[e^{i\psi_{xj}} - 2e^{i(2\psi_{xi}-\psi_{xj})} + e^{-i(2\psi_{xi}-3\psi_{xj})}\right]\Bigg\}, \end{aligned} \tag{27}$$

$$h_{11101} = h^*_{11011} = 0, \tag{28}$$

$$h_{30001} = h^*_{03001} = \sum \left\{ -\frac{i}{32}\bar{b}_{3i}\bar{b}_{2j}\beta_{xi}^{3/2}\beta_{xj}\left[e^{i3\psi_{xi}} - e^{i(\psi_{xi}+2\psi_{xj})}\right] - \frac{i}{16}\bar{b}_{3i}\bar{b}_{3j}\beta_{xi}\beta_{xj}^{3/2}\eta_{xi}\left[e^{i3\psi_{xj}} - e^{i(2\psi_{xi}+\psi_{xj})}\right]\right\}, \tag{29}$$

$$h_{20011} = h^*_{02101} = 0, \tag{30}$$

$$h_{20101} = h^*_{02011} = 0, \tag{31}$$

$$\begin{aligned} h_{10021} = h^*_{01201} = \sum \Bigg\{ & \frac{i}{32}\bar{b}_{3i}\bar{b}_{2j}\sqrt{\beta_{xi}}\beta_{xj}\beta_{yi}\left[e^{i(\psi_{xi}-2\psi_{yi})} - e^{-i(\psi_{xi}-2\psi_{xj}+2\psi_{yi})}\right] \\ & +\frac{i}{16}\bar{b}_{3i}\bar{b}_{2j}\sqrt{\beta_{xi}}\beta_{yi}\beta_{yj}\left[e^{i(\psi_{xi}-2\psi_{yi})} - e^{i(\psi_{xi}-2\psi_{yj})}\right] \\ & +\frac{i}{16}\bar{b}_{3i}\bar{b}_{3j}\beta_{xi}\sqrt{\beta_{xj}}\beta_{yj}\eta_{xi}\left[e^{i(\psi_{xj}-2\psi_{yj})} - e^{i(2\psi_{xi}-\psi_{xj}-2\psi_{yj})}\right] \\ & -\frac{i}{8}\bar{b}_{3i}\bar{b}_{3j}\sqrt{\beta_{xj}}\beta_{yi}\beta_{yj}\eta_{xi}\left[e^{i(\psi_{xj}-2\psi_{yi})} - e^{i(\psi_{xj}-2\psi_{yj})}\right]\Bigg\}, \end{aligned} \tag{32}$$

$$\begin{aligned} h_{10111} = h^*_{01111} = \sum \Bigg\{ & \frac{i}{16}\bar{b}_{3i}\bar{b}_{2j}\sqrt{\beta_{xi}}\beta_{xj}\beta_{yi}\left[e^{i\psi_{xi}} - e^{-i(\psi_{xi}-2\psi_{xj})}\right] \\ & +\frac{i}{16}\bar{b}_{3i}\bar{b}_{2j}\sqrt{\beta_{xi}}\beta_{yi}\beta_{yj}\left[e^{i(\psi_{xi}-2\psi_{yi}+2\psi_{yj})} - e^{i(\psi_{xi}+2\psi_{yi}-2\psi_{yj})}\right] \\ & +\frac{i}{8}\bar{b}_{3i}\bar{b}_{3j}\beta_{xi}\sqrt{\beta_{xj}}\beta_{yj}\eta_{xi}\left[e^{i\psi_{xj}} - e^{i(2\psi_{xi}-\psi_{xj})}\right] \\ & -\frac{i}{8}\bar{b}_{3i}\bar{b}_{3j}\sqrt{\beta_{xj}}\beta_{yi}\beta_{yj}\eta_{xi}\left[e^{i(\psi_{xj}-2\psi_{yi}+2\psi_{yj})} - e^{i(\psi_{xj}+2\psi_{yi}-2\psi_{yj})}\right]\Bigg\}, \end{aligned} \tag{33}$$

$$\begin{aligned} h_{10201} = h^*_{01021} = \sum \Bigg\{ & \frac{i}{32}\bar{b}_{3i}\bar{b}_{2j}\sqrt{\beta_{xi}}\beta_{xj}\beta_{yi}\left[e^{i(\psi_{xi}+2\psi_{yi})} - e^{-i(\psi_{xi}-2\psi_{xj}-2\psi_{yi})}\right] \\ & -\frac{i}{16}\bar{b}_{3i}\bar{b}_{2j}\sqrt{\beta_{xi}}\beta_{yi}\beta_{yj}\left[e^{i(\psi_{xi}+2\psi_{yi})} - e^{i(\psi_{xi}+2\psi_{yj})}\right] \\ & +\frac{i}{16}\bar{b}_{3i}\bar{b}_{3j}\beta_{xi}\sqrt{\beta_{xj}}\beta_{yj}\eta_{xi}\left[e^{i(\psi_{xj}+2\psi_{yj})} - e^{i(2\psi_{xi}-\psi_{xj}+2\psi_{yj})}\right] \\ & +\frac{i}{8}\bar{b}_{3i}\bar{b}_{3j}\sqrt{\beta_{xj}}\beta_{yi}\beta_{yj}\eta_{xi}\left[e^{i(\psi_{xj}+2\psi_{yi})} - e^{i(\psi_{xj}+2\psi_{yj})}\right]\Bigg\}, \end{aligned} \tag{34}$$

$$h_{00211} = h^*_{00121} = 0, \tag{35}$$

$$h_{00301} = h^*_{00031} = 0, \tag{36}$$

$$\begin{aligned} h_{11002} = \sum \Bigg\{ & \frac{i}{16}\beta_{xi}\beta_{xj}\left[(\bar{b}_{2i}\bar{b}_{2j} - 2\bar{b}_{3i}\bar{b}_{2j}\eta_{xi} + 4\bar{b}_{3i}\bar{b}_{3j}\eta_{xi}\eta_{xj})e^{i2(\psi_{xi}-\psi_{xj})} + 2\bar{b}_{3i}\bar{b}_{2j}\eta_{xi}e^{-i2(\psi_{xi}-\psi_{xj})}\right] \\ & +\frac{i}{8}\sqrt{\beta_{xi}}\beta_{xj}^{3/2}\eta_{xi}(\bar{b}_{3i}\eta_{xi} - \bar{b}_{2i})\bar{b}_{3j}\left[e^{i(\psi_{xi}-\psi_{xj})} - e^{-i(\psi_{xi}-\psi_{xj})}\right]\Bigg\}, \end{aligned} \tag{37}$$

$$\begin{aligned} h_{20002} = h^*_{02002} = \sum \Bigg\{ & \frac{i}{16}\beta_{xi}\beta_{xj}\left[(\bar{b}_{2i}\bar{b}_{2j} - 2\bar{b}_{3i}\bar{b}_{2j}\eta_{xi} + 4\bar{b}_{3i}\bar{b}_{3j}\eta_{xi}\eta_{xj})e^{i2\psi_{xi}} + 2\bar{b}_{3i}\bar{b}_{2j}\eta_{xi}e^{i2\psi_{xj}}\right] \\ & +\frac{i}{16}\sqrt{\beta_{xi}}\beta_{xj}^{3/2}\eta_{xi}(\bar{b}_{3i}\eta_{xi} - \bar{b}_{2i})\bar{b}_{3j}\left[e^{i(\psi_{xi}+\psi_{xj})} - e^{-i(\psi_{xi}-3\psi_{xj})}\right]\Bigg\}, \end{aligned} \tag{38}$$

$$h_{10012} = h^*_{01102} = 0, \tag{39}$$

$$h_{10102} = h^*_{01012} = 0, \tag{40}$$

$$h_{00112} = \sum \left\{ \frac{i}{16}\beta_{yi}\beta_{yj} \left[(\bar{b}_{2i}\bar{b}_{2j} - 2\bar{b}_{3i}\bar{b}_{2j}\eta_{xi} + 4\bar{b}_{3i}\bar{b}_{3j}\eta_{xi}\eta_{xj})e^{i2(\psi_{yi}-\psi_{yj})} + 2\bar{b}_{3i}\bar{b}_{2j}\eta_{xi}e^{-i2(\psi_{yi}-\psi_{yj})} \right] \right.$$
$$\left. -\frac{i}{8}\sqrt{\beta_{xi}\beta_{xj}}\beta_{yj}\eta_{xi}(\bar{b}_{3i}\eta_{xi} - \bar{b}_{2i})\bar{b}_{3j} \left[e^{i(\psi_{xi}-\psi_{xj})} - e^{-i(\psi_{xi}-\psi_{xj})} \right] \right\}, \tag{41}$$

$$h_{00202} = h^*_{00022} = \sum \left\{ \frac{i}{16}\beta_{yi}\beta_{yj} \left[(\bar{b}_{2i}\bar{b}_{2j} - 2\bar{b}_{3i}\bar{b}_{2j}\eta_{xi} + 4\bar{b}_{3i}\bar{b}_{3j}\eta_{xi}\eta_{xj})e^{i2\psi_{yi}} + 2\bar{b}_{3i}\bar{b}_{2j}\eta_{xi}e^{i2\psi_{yj}} \right] \right.$$
$$\left. -\frac{i}{16}\sqrt{\beta_{xi}\beta_{xj}}\beta_{yj}\eta_{xi}(\bar{b}_{3i}\eta_{xi} - \bar{b}_{2i})\bar{b}_{3j} \left[e^{i(\psi_{xi}-\psi_{xj}+2\psi_{yj})} - e^{-i(\psi_{xi}-\psi_{xj}-2\psi_{yj})} \right] \right\}, \tag{42}$$

$$h_{10003} = h^*_{01003} = \sum \left\{ \frac{i}{4}\bar{b}_{3i}\bar{b}_{2j}\beta_{xi}\sqrt{\beta_{xj}}\eta_{xi}\eta_{xj} \left[e^{i\psi_{xj}} - e^{i(2\psi_{xi}-\psi_{xj})} \right] \right.$$
$$\left. +\frac{i}{8}\sqrt{\beta_{xi}}\beta_{xj}\eta_{xi} \left[\bar{b}_{2i}\bar{b}_{2j} - \bar{b}_{3i}\eta_{xi}(\bar{b}_{2j} - 2\bar{b}_{3j}\eta_{xj}) \right] \left[e^{i\psi_{xi}} - e^{-i(\psi_{xi}-2\psi_{xj})} \right] \right\}, \tag{43}$$

$$h_{00103} = h^*_{00013} = 0, \quad \text{and} \tag{44}$$

$$h_{00004} = \sum \frac{i}{4}\sqrt{\beta_{xi}\beta_{xj}}\eta_{xi}\eta_{xj} \left\{ \left[\bar{b}_{2i}\bar{b}_{2j} - \bar{b}_{3i}\eta_{xi}(\bar{b}_{2j} - \bar{b}_{3j}\eta_{xj}) \right] e^{i(\psi_{xi}-\psi_{xj})} + \bar{b}_{3i}\bar{b}_{2j}\eta_{xi}e^{-i(\psi_{xi}-\psi_{xj})} \right\}. \tag{45}$$

Here the coefficients $\bar{b}_2 = b_2 L = K_1 L$ and $\bar{b}_3 = b_3 L = K_2 L/2$, i.e., the integrated field strength (K_1 and K_2 are quadrupole and sextupole coefficients used in ELEGANT and MAD, and L is magnet length). The summation means

$$\sum f(i,j) \equiv \sum_{j>i} [f(i,j) - f(j,i)] = \left(\sum_{j>i} - \sum_{i>j} \right) f(i,j). \tag{46}$$

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[1] C.-x. Wang and A. Chao, "Notes on Lie algebraic analysis of achromats," SLAC note AP-100 (1995).

[2] A. Dragt, in *Accelerator Physics and Engineering*, edited by A.W. Chao and M. Tigner, Eq. (33) on p. 78. Singapore: World Scientific Publishing, 1999.

[3] J. Bentsson, "The sextupole scheme for the Swiss light source (SLS): an analytic approach," SLS Note 9/97 (1997).