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\begin{abstract}
In this paper a framework for developing a coherent theory of mathematics and physics together is described. The main and possibly defining characteristic of such a theory is discussed: the theory must maximally describe its own validity and completeness, and must be maximally valid and complete. Definitions of validity and completeness are based on those used in mathematical logic. A coherent theory is universally applicable, so its domain includes intelligent physical systems that test the validity of the theory. Anthropic aspects of a coherent theory are discussed. It is suggested that the basic properties of the physical universe are entwined with and emerge from such a theory. It is even possible that the condition that there exists a coherent theory satisfying the maximal validity and completeness requirement is so restrictive that there is just one theory and one physical universe satisfying the theory. Other aspects include the proof that the meaning content of a coherent theory (or any axiomatizable theory) is independent of the information content of the theory. The observation that language is physical is discussed. All symbols and words of any language necessarily have physical representations as states of physical systems that are in the domain of a coherent theory. This is an important property not enjoyed by any purely mathematical theory. An example of a physical representation of language is described in the appendix.
\end{abstract}

\section{Introduction}

It is perhaps obvious that there is a very close relationship between physics and mathematics. That theoretical physics is essentially mathematical in nature can be seen by a cursory examination of most any book on theoretical physics. Also theoretical physics is used to generate predictions that can be affirmed or refuted by experiment. Often these predictions are generated as outcomes of complex computations. If these outcomes are numbers then they are compared with the numerical outcomes of relevant experiments. The validity of a physical theory is based on many such comparisons between theory and experiment. Agreement constitutes support for the theory. Disagreement between theoretical predictions and experiment erodes support for the theory and, for crucial experiments, may result in the theory being abandoned.

Yet it also seems that there is a disconnect between physics and mathematics. One way to see the problem is to note that, from a foundational point of view, physics takes mathematics for granted. In many ways theoretical physics treats mathematics much like a warehouse of different consistent axiom systems each with their set of theorems. If a system needed by physics has been studied, it is taken from the warehouse, existing theorems and results are used, and, if needed, new theorems are proved. If theoretical physics needs a system which has not been invented, it is created as a new system. Then the needed theorems are proved based on the axioms of the new system.

The problem here is that physics and mathematics are considered as separate disciplines. The possibility that they might be part of a larger coherent theory of mathematics and physics together is not much discussed. For example basic aspects such as truth, validity, consistency, and provability are described in detail in mathematical logic which is the study of axiom systems and their models \cite{Shoenfield}. The possibility that how these concepts are described or defined may affect their use in physics, and may also even influence what is true in physics at a very basic level has not been considered. (A very preliminary attempt to see how these concepts might be used in quantum mechanics is made in \cite{BenDTVQM}.)

The situation in mathematics is different. Here the problem is that most work in mathematics and mathematical logic is purely abstract with little attention paid to foundational aspects of physics. The facts that mathematical reasoning is carried out by physical systems subject to physical laws, and symbols, words, and formulas in any language, formal or not, are physical systems in different states, is, for the most part, ignored. In some ways the various constructivist interpretations of mathematics, ranging from extreme intuitionism \cite{Heyting} to more moderate views \cite{Bishop,Beeson} (see also \cite{Frankel}) do acknowledge this problem.

However most mathematicians and physicists ignore any limitations imposed by constructivist viewpoints. Their activities appear to be based implicitly on the ideal or Platonic viewpoint of mathematical existence, i.e. that mathematical entities and statements have an ideal existence and truth status independent of any physical limitations \cite{Penrose} or an observers knowledge of them \cite{Hersh,Kline}. The "luscious jungle flora" \cite{Frankel} aspect of this view of mathematical existence compared to the more ascetic landscape \cite{Frankel} of more constructivist views is hard to resist.

However, this viewpoint has the problem that one must face the existence of two types of objects. There are the ideal mathematical objects that exist outside space-time and the physical objects that exist inside of and influence the properties of space-time. The existence of two types of objects that appear to be unrelated yet are also closely related is quite unsatisfactory.

Another approach to mathematics is that of the formalist school \cite{Hersh,Kline}, Here mathematics is considered to be in essence like a game in which symbol strings (statements or formulas) are manipulated according to well defined rules. The goal is the rigorous proof of theorems. Mathematical entities have no independent reality status or meaning.

In one sense the formalist school is related to physics in that provability and computability are closely related. In work on computability and computational complexity \cite{Chaitin,Papadimitriou} it is clearly realized that computability is related closely to what can be carried out (in an ideal sense) on a physical computer. Yet, as has been noted \cite{DeutschMLQ}, the exact nature of the relationship between computations carried out by real physical computers and abstract ideal computers, such as Turing machines, is not clear.

The influence of physics on mathematics is perhaps most apparent in recent work on quantum information theory and quantum computing. Here it has been shown \cite{Shor,Grover} that there exist problems that can in principle be solved more efficiently on a quantum computer than by any known classical computational algorithm. Also the increased efficiency of simulation of physical quantum systems on quantum computers \cite{Feynman,Zalka,Wiesner,Abrams} compared to simulation on classical systems is relevant to these considerations.

The problems on the relationship between physics and mathematics have been considered by others. In his insistence that "Information is Physical" Landauer \cite{LandauerIP} also recognizes the importance of this relationship. His reference to the fact that, according to Bridgman, mathematics should be confined to what are in essence programmable sequences of operations, or that mathematics is empirical \cite{Bridgman}, supports this viewpoint. Similar views on the need for an operational characterization of physical and set theoretic entities has been expressed \cite{Svozil}.

Other attempts to show the importance of physics on the foundations of mathematics include work on randomness \cite{BenRAN} and on quantum set theory \cite{Finkelstein,Takeuti} (see also \cite{Nishimura}). Recent work on the relationship between the Riemann hypothesis and aspects of quantum mechanics \cite{Odlyzko,Crandall} and relativity \cite{Okubo}, and efforts to connect quantum mechanics and quantum computing with logic should be noted \cite{Ozhigov,Buhrman,Schmidhuber} along with efforts to connect mathematical logic with physics \cite{Tegmark,Spector,Foschini}.

In spite of this progress both the lack of and a need for a coherent theory of mathematics and physics together remain. One view of this is expressed by the title of a paper by Wigner \cite{Wigner} "On the unreasonable effectiveness of mathematics in the natural sciences". One does not know why mathematics is so effective and an explanation is needed. Even though the paper was published in 1960, it is still relevant today. A related question, "Why is the physical world so comprehensible?" \cite{Davies} also needs to be answered.

The purpose of this paper is to examine in more detail some aspects of the relationship between physics and mathematics. The goal is to work towards the development of a coherent theory of mathematics and physics. In particular the main (and possibly defining) characteristic of such a theory is proposed and discussed. This is that the theory should {\em maximally describe its own validity and completeness}, and it should {\em be maximally valid and complete}. It is also possible that this condition is so restrictive that there is only one theory that satisfies this condition. Furthermore, it may also be the case that the uniqueness of the theory implies that there is only one physical universe that satisfies the theory.

These ideas will be discussed in the following sections. The approach suggested here is to work towards a coherent theory of mathematics and physics by combining mathematical logical concepts with quantum mechanics or some suitable generalization such as quantum field theory. As such the purpose of this paper is to provide a background or framework of ideas which will guide future work on the construction of a coherent theory of mathematics and physics.

Since mathematical logic deals with axiomatizable theories based on formal languages and their models, theories described here will also be considered as formal axiomatic systems. This does {\it not} mean that physics and mathematics should adopt formal methods in the proofs and derivations of new results and theorems. The reason for taking this approach is that semantic and syntactic aspects or theories and their interpretations are clearly separated. Also the distinction between the domain of discourse of the theory and metatheoretical aspects is made clear.

It should be noted that there are many axiomatizations of physical theories in the literature. In particular axiomatizations of quantum mechanics and quantum field theory have been much studied. These include algebraic approaches \cite{Mackey,Haag}, quantum logic approaches \cite{BirkhfVnN,Jauch} and others \cite{Hardy}. These axiomatizations which are often quite mathematical and rigorous, could be axiomatized in the formal sense discussed here. However, in common with most axiomatizations of mathematical and physical theories, this is not done as it is not necessary for the purposes of the studies.

\section{A Coherent Theory of Mathematics and Physics}
\label{CTMP} The basic idea is that a coherent theory of mathematics and physics includes a coherent description of both the mathematical and physical components of the universe. The theory must also satisfy a basic and possibly defining requirement. That is, the coherent theory must be able to maximally describe its own validity and completeness and it must be maximally valid and complete.

If quantum mechanics, or some suitable generalization such as quantum field theory, is taken as the physical component, then a coherent theory of mathematics and quantum mechanics must satisfy the basic requirement. It must maximally describe its own validity and completeness and it must be maximally valid and complete.

\subsection{Validity and Completeness in the Requirement}

It is expected that the definitions of validity and completeness for the coherent theory will be similar to those used in mathematical logic \cite{Smullyan,Shoenfield}. A formal theory is valid if all formulas or expressions in the language of the theory that are theorems are true. The theory is complete if for each closed formula (an expression with no free variables) either it or its negation, but not both, is a theorem. Group theory, the theory of real numbers, and nonatomic Boolean algebra are examples of complete theories. As shown by G\"{o}del \cite{Godel} there are also incomplete theories. Examples include arithmetic, set theory, and any other theory that includes arithmetic.

Here the definitions of validity and completeness must take into account that one is dealing with a theory that includes both mathematics and physics. An informal definition of validity is that the theory is valid if each property of a physical system in the domain of the theory, that is predicted by the theory and is capable of experimental verification or refutation, is in fact verified by experiment.

The definition of validity says that for each physical property of a system, if there is a theorem of the theory stating that the system has the property, then it must be true, provided it can be experimentally verified. It follows from this\footnote{This is based on the logic of if-then statements which are true if the "if" part is false.} that a theory is valid if it makes no predictions at all. Less extreme cases are included in which a theory makes very few predictions, which must be true.

These possibilities can be removed by requiring that the theory be maximally complete. Here one follows mathematical logic by defining a coherent theory to be complete if for each property of a physical system in the theory domain, either it or its negation, but not both, is predicted by the theory and is capable of experimental test.

As is well known, quantum mechanics is not complete in this sense. For instance if one assumes that single measurements of observables on quantum systems are properties of physical systems, then incompleteness follows from the predictability of expectation values only of observables; individual measurement outcomes are not predictable. Also there may well be other complex properties, including those that are self referential and similar to those used by G\"{o}del in his proof of the first incompleteness theorem \cite{Godel}, that cannot be predicted or measured. This is the reason for the qualification of {\em maximal} for validity and completeness in the requirement for the coherent theory of mathematics and physics.

In the definitions of validity and completeness, physical procedures enter in two places: the determining of which statements are theorems and thus are testable physical properties, and validity testing of these predictable properties. For theorems physical procedures are important to the extent that theorem proving can be implemented on a real physical computer. Methods for implementing this are based on the arithmetization of proofs by use of G\"{o}del maps and the requirement that arithmetic and other operations can be implemented on a physical computer.

For physical theories, and a coherent theory of physics and mathematics, physical procedures also enter with the validity requirement that there exists an experimental test for any predictable property. This creates a problem in that predictability of a property for a physical system does not guarantee that there exists a physical procedure for determining whether the system has or does not have the property.

The problem here is that there is no way so far to define physical implementability for procedures. This includes procedures for preparing systems in different quantum states and procedures for measuring observables, represented by self adjoint operators in an algebra of operators. It is clear that for many states of complex quantum systems and for many observables it is very unlikely that there exist efficiently implementable physical procedures for preparing the states and measuring the observables.

Examples of these states include complex entangled states of multicomponent systems of the type studied by Bennett et al \cite{Bennettetal}. Examples of observables include projection operators on these entangled states. A related example is based on the observation that efficient physical implementability is not preserved under arbitrary unitary transformations of self adoint operators \cite{BenEIPSRN}. Thus if the observable $ \check{O}$ is efficiently implementable it does not follow that $U\check{O}U^{\dagger}$ is implementable for arbitrary unitary U. It has also been noted that there is a problem in determining exactly which logical procedures or algorithms are physically implementable \cite{DeutschMLQ}. This problem is especially relevant for quantum computer algorithms as it is not at all clear which are efficiently physically implementable and which are not.

The requirement that a theory maximally describe its own validity and completeness means that there are one or more formulas in the language of the coherent theory that can be interpreted to mean that the theory is maximally valid and complete. These formulas are likely to be very complex as they must somehow express the concepts of truth and provability for the theory. As such these formulas may not be theorems of the theory.

To see why, it is easier to consider an alternate form of the requirement. This is that the coherent theory of mathematics and quantum mechanics must be maximally self consistent. That is, the theory must refer to its own consistency as much as is possible and it must be consistent.

The usefulness of this form of the requirement is based on the properties of consistency. An axiom system is defined to be consistent if not all formulas are theorems, or, equivalently, if it has a model. A model is a universe of objects, relations, and functions, etc., in which relations are either true or false and the various symbols and formulas of the theory obtain meaning through their interpretation as objects or relations in the model.

These definitions show the importance of consistency as inconsistent axiom systems are useless in that any formula, as a statement in the language on which the axiom system is based, is a theorem. Also all formulas based on an inconsistent axiom system are meaningless as they have no interpretation in a model.

G\"{o}del's second incompleteness theorem \cite{Shoenfield,Smullyan,Godel} is directly applicable to the problem at hand because it says that if a theory is consistent and is strong enough to express consistency of the theory, then the formula expressing consistency is not a theorem of the theory. Also the theorem applies to any extension of the theory in which the formula expressing the consistence of the original theory is provable.

Based on this, any formula in the language of the coherent theory of physics and mathematics that expresses the consistency of the theory is probably not a theorem of the theory. It is also likely to be the case that any formula expressing the validity and completeness of the theory is also not a theorem of the theory. However the lack of theoremhood for these formulas are not theorems says nothing about whether they are true or false. This is taken care of by the second part of the requirement; namely, that a coherent theory be maximally valid and complete. This part includes the requirement that the formulas expressing the validity and completeness of the coherent theory are true.

So far the requirement has been expressed in two forms, one using consistency and the other using validity and completeness. At present it is not clear which form is preferable. It is suspected, based on a very simple model \cite{BenDTVQM,BenUMLCQM}, that the form expressed using validity and completeness is stronger. For this reason the form of the requirement, as originally stated in terms of validity and completeness, will be used in the rest of the paper.

\subsection{Universal Applicability} \label{UA}

Another property that a coherent theory of mathematics and physics should have is that it is universally applicable. This includes both the physical and mathematical components of the theory. Since quantum mechanics or some generalization is, at present, the physical theory assumed to be universally applicable, it is expected that a coherent theory of mathematics and physics is a coherent theory of mathematics and quantum mechanics that maximally describes its own validity and completeness and must be maximally valid and complete.

Universal applicability of the mathematical component is taken into account by including mathematical logical concepts with quantum mechanics. Since mathematical logic includes the study of mathematical systems as axiomatizable systems and their interpretations, universal applicability means that all the mathematical systems used in quantum mechanics and its extensions that are in use or may be used in the future are included. Also included are basic properties of physical and mathematical theories, such as truth, validity, completeness, consistency, and provability, that are treated in mathematical logic. A coherent theory would be expected to combine these concepts with quantum mechanics.

At this point it is not clear how to exactly define universal applicability for the physical component of the theory, or whether one should even separate the theory into mathematical and physical components. In the case of quantum mechanics, one view \cite{Tegmark1} is that defining universal applicability to include all physical systems means that one must accept the Everett interpretation \cite{Everett,Wheeler1} of quantum mechanics. This interpretation assumes that the whole universe is described by a quantum state evolving according to quantum dynamical laws.

One way to avoid this may be to assume that universal applicability means that the theory is applicable only to systems that are subsystems or part of other larger systems that are in turn subsystems of other still larger systems.\footnote{A more precise statement of this might be: (1) If the theory is applicable to subsystem A, then there exist many subsystems B that contain A and to which the theory is applicable. (2) There are many subsystems A to which the theory is applicable. Furthermore the definition of "many subsystems" must be sufficiently broad to include all subsystems accessible to state preparation and experiment.} This includes both open and closed subsystems including those that may be isolated for a period of time.

The exact definition is probably best left to further development of a coherent theory. However the definition should be such to include subsystems described by a finite number of degrees of freedom and many systems, such as quantum fields, that are described by an infinite number of degrees of freedom.

Based on this condition, universal applicability supports the requirement that the coherent theory maximally describe its own validity and completeness. To see this one notes that validation of a physical theory includes the comparison of theoretical predictions with the results of experiment. In general the theoretical calculations and predictions are made by computers and experiments are carried out by robots or intelligent systems using different pieces of equipment. The validation process is carried out by intelligent systems. If the coherent theory is universally applicable then the dynamics of the computers, robots, experimental equipment, and intelligent systems carrying out the validation must be described by the physical dynamical laws of the theory.

The importance of the requirement of universal applicability is that the properties and dynamics of the systems implementing the validation of the theory are included in the domain of applicability of the theory to the maximum extent possible. In the case of quantum mechanics or some suitable generalization this means that computers and robots are quantum mechanical systems with dynamics described by quantum dynamical laws. This holds for both microscopic systems, such as quantum computers and quantum robots \cite{BenQR} and macroscopic systems such as classical computers and robots which are in very wide use at present.

It also follows that intelligent systems are quantum mechanical systems. The observation that the only known examples (including the readers of this paper) of intelligent systems are macroscopic, with about 10^{25} degrees of freedom, does not contradict the quantum mechanical nature of these systems. This may be a reflection of the possibility that a necessary requirement for a quantum system to be intelligent is that it is macroscopic. However, whether this is or is not the case, is not known at present.

That intelligent observers are both conscious self aware systems and quantum systems has been the basis for much discussion on consciousness in quantum mechanics \cite{Penrose,Stapp,Squires,Page}. Included are discussions on interactions between two quantum observers \cite{Wigner1,Albert}. These avenues will not be pursued here as they do not seem to be the best way to progress towards developing a coherent theory of mathematics and physics.

It follows that the dynamics of the quantum systems carrying out the validation of quantum mechanics must be described by quantum dynamical laws. Thus quantum mechanics must be able to describe the dynamics of its own validation process. However validation of a theory involves more than just describing the dynamics of the systems carrying out the validation. Validation includes the association of meaning to the results of theoretical derivations and computations carried out by quantum systems (as computers). Meaning must also be associated to the results of carrying out experiments by quantum systems (as robots or intelligent systems).

This association of meaning to the results of quantum processes is essential. It is basic to determining which processes constitute valid procedures, either computational or experimental. These processes are a very small fraction of the totality of all processes that can be carried out, most of which have no meaning at all. They are neither computations or experiments.

This association of meaning includes such essentials as the (nontrivial) assignment of numbers to the results of both computational and experimental process. A computation process or an experiment that halts produces a complex physical system in a particular physical state. What numbers, if any that are associated to the states depend on the meaning and validity of the process \cite{BenRNQM,BenRNQMALG}. That is the process must be a valid computational or experimental procedure. If it is valid then one must know the property to which it refers. This is needed to know the association between theoretical computations and experimental procedures.

For example in quantum mechanics for some observable O and state Ψ one must be able to determine which of the valid computation procedures is a computation of the expectation value $\langle \Psi|O|\Psi\rangle$. One must also know which of the valid experimental procedures corresponds to a measurement of this expectation value. As is well known the experiment must in general be repeated may times to generate the expectation value as a limit as $n\rightarrow \infty$ of the average of the first n repetitions of the experiment. Association of meaning to these procedures also includes all the components involved in determining that appropriate limits exist for both the computation procedures and experimental procedures.

A coherent theory of mathematics and quantum mechanics must be able to express as much of this meaning as is possible. Thus not only must it be able to express the dynamics of its own validation, but it must be able to express the meaning associations described above to the maximum extent possible. Thus it must be able to maximally describe its own validation.

A similar argument applies to maximal completeness. That is, there must be a sense in which a coherent theory includes all properties that are predictable and capable of experimental test. Of course the problem here lies in the exact meaning of "all properties that $\cdots$". One may hope and expect that a coherent theory would be able to express to the maximum extent possible the meaning of "all properties that $\cdots$". And it should also be able to express the condition that it is also maximally complete.

The possibility that there may be no single theory that is maximally complete should also be considered. Instead there may be a nonterminating sequence of theories of increasing completeness. This possibility will be discussed more later on.

These arguments form the basis for the requirement that the coherent theory maximally describe its own validation and completeness. However, being able to generate such a description does not guarantee that the coherent theory {\it is} valid and complete. A theory may be interpreted to express that it has some property, but it does not follow that it actually has that property. This possibility is excluded for the coherent theory by also requiring that it is maximally valid and complete.

\subsection{The Coherent Theory and the Strong Anthropic Principle}

The conditions that a coherent theory include both physics and mathematics and that it satisfy the requirement of maximal description of its validity and completeness and be maximally valid and complete, suggest that there may be a very close relation between the theory and the basic properties of the physical universe. It may be the case that at a very basic level the basic properties of the physical universe are entwined with and may even be determined by a coherent theory that satisfies the requirements.

Examples of such basic properties that may emerge from or be determined by the coherent theory include such aspects as the reason for three space and one time dimension (See Tegmark \cite{Tegmark} for another viewpoint), the strengths and reason for existence of the four basic forces, why quantum mechanics is the valid physical theory, etc.. Even if few or none of these properties are determined, one may hope that the theory will shed new light on already explained basic properties.

These possibilities suggest that a coherent theory with the requirement is related to the strong anthropic principle \cite{BarTip,Hogan,Greenstein}. This principle can be stated in different ways. One statement is that "The basic properties of the universe must be such that [intelligent]life can develop" \cite{BarTip}. Wheeler's interpretation as quoted by Barrow and Tipler \cite{BarTip} is that "Observers are necessary to bring the universe into being". A stronger statement is the final anthropic principle \cite{BarTip} "Intelligent information processing must occur and never die out".

The relation between this principle and a coherent theory can be seen by recasting the statement of the maximal validity and completeness requirement into an existence statement or condition: {\em There exists a coherent theory of physics and mathematics that maximally describes its own validity and completeness and is maximally valid and complete}. In this case the basic properties of the physical universe emerge from or are a consequence of the existence statement. That is, the basic properties of the physical universe must be such that the existence statement is true.

Another way to state this is that the basic properties of the physical universe must be such that a coherent theory is creatable. Since intelligent beings are necessary to create such a theory, it follows that the basic properties of the physical universe must be such as to make it possible for intelligent beings to exist. Since the intelligent beings, as physical systems, are part of the physical universe, the theory must, in some sense, also refer to its own creatability.

None of this implies that intelligent beings must exist, only that it must be possible for them to exist. Of course existence of intelligent beings is a necessary condition for the actual creation of such a coherent theory.

\subsection{On the Possible Uniqueness of the Coherent Theory}
The requirement that a coherent theory of mathematics and physics maximally describe its own validity and completeness and be maximally valid and complete would seem to be quite restrictive. Indeed one may speculate that the condition is so restrictive that there is just one such theory.

One reason this might be the case is that if there were several different coherent theories each satisfying the requirement, then there would be several different physical universes, with the basic physical properties of each universe determined by one of the theories. Yet we are aware of just one physical universe, the one we inhabit, with the basic properties determined by both physical theory and experiment. It follows that if the basic properties of the physical universe are determined by a coherent theory satisfying the requirement, then the existence of just one physical universe implies that there is just one coherent theory satisfying the requirement. (Here coherent theories that may differ in some manner but determine the same physical properties of a universe are identified as one theory.)

Another aspect that may restrict the number of acceptable coherent theories is the emphasis that the theory {\em maximally} describe its own validity and completeness and that it is {\em maximally} valid and complete. There may well be many coherent theories that partly describe their own validity and completeness and are partly valid and complete. If such theories exist, they would be eliminated by the requirements of maximal description of their own validity and completeness and that they are maximally valid and complete.

Viewed from this uniqueness perspective, the basic statement that there exists just one coherent theory of physics and mathematics that maximally describes its own validity and completeness and is maximally valid and complete becomes a quite powerful axiom. The reason is that it can be used with the arguments given above to obtain the result that there is just one physical universe with basic properties determined by the unique theory. And this should be our universe.

If this line of reasoning is indeed valid, then it would be very satisfying as it answers the question, "Why does our physical universe have the properties it does?". Answer: The physical universe could not be otherwise as it is the only one whose properties emerge from or are determined by the coherent theory. No other universe is possible because there is just one coherent theory satisfying the maximality requirement and each such theory is associated with just one physical universe.

At present this argument, although appealing, must be regarded as speculation. Whether it is true or not must await development of a coherent theory of physics and mathematics, if such is even possible.

\subsection{Emergence of the Basic Properties of the Physical Universe} At present the main approach to physics seems to be that one assumes implicitly a physical universe whose basic properties exist independent of and a priori to a theoretical description, supported by experiment, of the universe. This is implied by reference to experiments as "discovering properties of nature". A priori, independent existence of the physical universe is also implied in the expression used above "theoretical description, supported by experiment, of the universe".

The approach to mathematics is much more variable as there are many different interpretations of the meaning of existence in mathematics \cite{Hersh,Kline,MarMyc}. However, the Platonic viewpoint that is widely accepted, at least implicitly, is that mathematical objects exist a priori to and independent of a theoretical description of them with their properties to be discovered by mathematical research.

Here it is suggested that one should regard the basic properties of the physical and mathematical universes as very much entwined with a coherent theory of mathematics and physics. Neither the mathematical universe, physical universe, nor the coherent theory should be considered to be a priori and independent of the other two components. The basic properties of all three components should be considered to be emergent together and mutually determined.

This means that, for the relation between the physical universe and the coherent theory, the basic physical aspects of the physical universe should be considered to emerge from and be determined by the basic properties of a coherent theory of physics and mathematics. Also the basic properties of a coherent theory should, in turn, emerge from and be determined by the basic properties of the physical universe.

It must be strongly emphasized that the emergence noted above does {\em not} mean that there is any arbitrariness to the basic physical properties and that an observer can choose them as he pleases. Rather the viewpoint taken here suggests that a coherent theory that maximally describes its own validity and completeness and is maximally valid and complete, is also maximally objective. The reason is that a maximally self referential theory refers to as much of its own consistency, validity, and completeness as is possible, and the role of an observer or intelligent being is thereby minimized in determining the basic properties of the theory. In this case the basic properties of the universe as described by a coherent theory must appear to any observer to be objective and real and maximally independent of the existence and activities of an observer. That is what one means by objectivity.

In one sense the idea of the emergence of basic physical properties of the universe is already in use. This is based on the condition that the more fundamental properties of the physical universe require many layers of theory supported by experiment to give them meaning. Their reality status is more indirect as it depends on many layers of theory supported by experiment.

For example the existence and properties of atoms is indirect in that it is based on all the experimental support for the many theoretical predictions based on the assumed existence and properties of atoms. One does not directly observe individual atoms. Pictures of individual atoms taken with an electron microscope depend on many layers of theory and experiment to determine that a complex physical system is an electron microscope and that the output patterns of light and dark shown on film or a screen are not meaningless but have meaning as pictures of individual atoms.

The physical reality and properties of more fundamental systems, such as quarks and gluons, are even more indirect than for atoms and depend on more intervening layers of theory and experiment. The same holds for neutrinos as fundamental systems whose reality status and properties are quite indirect. Experimental support for the existence of these particles depends on the layers of theory, which may include quantum electrodynamics, and all the supporting experiments needed to describe the proper functioning of large particle detectors and assigning meaning to the output of the detectors.

A similar situation exists for large, far away objects such as quasars. The reality status and physical properties of these systems are based on the theories of relativity and interactions of electromagnetic fields, etc.. These are needed to interpret the observations made using telescopes and to describe the proper functioning of telescopes and other equipment used.

On the other hand the reality status and some properties of other physical systems require little or no theoretical or experimental support. For example, the existence and hardness of rocks or the existence of the sun and the facts that it is hot, bright and round, are directly observed properties. Little theory with supporting experiment is needed to make these observations. Other properties of these objects are more indirect. An example is the description of the sun as a gravitating body generating energy by thermonuclear fusion of hydrogen.

It should be emphasized that none of the above implies that systems such as quarks, atoms, and quasars are any less real and objective than are rocks and the sun. Rather the point is that the reality of their existence and their properties are more indirect and depends on more intervening
layers of theory and experiment than is the case for rocks and the sun. Also the reality of all the properties of quarks, atoms and quasars, is indirect in its dependence on layers of theory and supporting experiment. For nearby moderate sized objects some of the properties are quite direct and some are more indirect. For example, as noted above, direct properties of the sun are that it is hot bright and round. Indirect properties include the source of its energy.

It is necessary to emphasize the importance of the intervening layers of theory and experiment needed to support the proper functioning and interpretation of complex equipment. Since most equipment involves the electromagnetic interactions between systems or between fields and systems, the theory of these interactions must be well understood to ensure that a given physical system is a properly functioning piece of equipment. This is needed to to ensure that certain properties of the system represent output and that the output has meaning.

\section{Basic Mathematical Properties and a Coherent Theory}

So far relatively little has been said about basic mathematical properties and how they might be entwined with basic physical properties and a coherent theory of mathematics and physics. Here by basic mathematical properties is meant those properties common to all theories, mathematical and physical, that are discussed in mathematical logic. This includes the properties of truth, validity, consistency, completeness, and provability and the condition that all theories are considered to be axiomatizable systems where the axioms are designated expressions in some language suitable for expressing the various statements of the theory.

Another basic property, discussed explicitly in mathematical logic, is the distinction between syntactical and semantic aspects of a theory \cite{Shoenfield}. Syntactic aspects are the purely formal aspects of a language and its expressions. Included are definitions of different symbols as variables, constants, function and relation symbols of a language, and definitions of symbol strings as terms and formulas or statements in the language. Also provability, completeness, and consistency have syntactical definitions.

Semantic aspects of a theory are those associated with the meaning given to the symbols and expressions of a language through interpretations or maps of the symbols and expressions to informal or intuitive model universes. Truth is taken to be an informal intuitive concept, and validity is defined relative to this concept.

These two aspects must be connected, and part of mathematical logic is concerned with theorems that relate syntactic and semantic aspects of an axiomatizable theory \cite{Shoenfield,MarMyc}. Included is G\"{o}dels completeness theorem for first order logic. This theorem says that a formula is a theorem of the theory if and only if it is valid (i.e. true in all models of the theory)\footnote{A model of a theory is a universe of objects in which the axioms of the theory are true.}. Another theorem states that an axiom system is consistent if and only if it has a model.

Entwining of these properties with those of a coherent theory of mathematics and physics results from the assumption that, in common with other theories, the statements of a coherent theory are expressible in some language with some of the statements corresponding to axioms of the theory. If the theory is only partially axiomatizable or is not axiomatizable, then this should be discovered in future work towards developing such a theory.

A coherent theory is different from purely mathematical theories in that models include physical universes. Since it is assumed here that the theory is axiomatizable in first order logic, the various theorems relating syntactic and semantic properties would be expected to hold. In particular G\"{o}del's completeness theorem relates the notion of provability or theoremhood to physical truth. That is, a coherent theory is valid if all predictable properties of physical systems, as theorems of the coherent theory, are true, at least in the universe we inhabit and in any others, if they exist \cite{Deutsch,Tegmark}.

Another property that is important for a coherent theory, and for other mathematical and physical theories is the condition that it must be physically possible to make theoretical predictions. The truth of this property, which is taken for granted, is supported by the wide use of computers in mathematics and physics to make computations whose results (numerical and otherwise) are used as predictions of physical or mathematical properties.

Yet this fact is quite nontrivial and involves many assumptions. Included is the requirement that there exist physical states of systems that represent numbers. As discussed elsewhere in detail \cite{BenRNQM,BenRNQMALG,BenEIPSRN}, the meaning of this for natural numbers is that there exist physical systems with states on which the basic arithmetic operations, with properties determined by the axioms or arithmetic or number theory, are efficiently physically implementable. Similar conditions were described for the other types of numbers, the integers and rational numbers. Although the emphasis of the discussion was for microscopic or quantum systems, the discussion holds also for macroscopic systems.

The importance of this is that if it were not possible to represent numbers by states of physical systems, then computers would not exist and it would be impossible to carry out computations. Predictions of properties, other than the most elementary and direct, would not be possible as there would exist no means to calculate or determine the property. Physics would not exist and it is doubtful that intelligent systems would exist as they can also carry out computations and make predictions.

These arguments show explicitly the interrelationship between basic mathematical properties and physical systems. Axioms of basic theories of different types of numbers are used as part of the conditions that must be satisfied so that physical systems admit states representing numbers. Also since a coherent theory of physics and mathematics is assumed to be axiomatizable, the well developed mathematical logical theorems and properties of this type of theory should apply.

\section{Language is Physical}

Another important aspect is based on the observation that the coherent theory, or any theory, must be expressed in some language. All languages in existence, formal or informal, such as English, have the property that they are based on the combination of symbols into strings of symbols or expressions. For the discussion here it is immaterial whether a symbol string is or is not also a string of words.

The point to note is that all symbols of a language are necessarily represented by physical systems in different states where the different states correspond to different symbols. This is the case for printed text, or for modulated waves moving through some medium as is the case for spoken language, or for language transmitted optically by use of photons.

As an example of this representation, consider the text of this paper. Each letter, word, paragraph, etc. is represented by physical systems in different physical states. This is the case whether the paper appears as printed material on pages of paper, patterns of light and dark regions on a computer screen, or as time variations in phase and amplitude of sound waves as when one speaks or delivers a lecture. It also applies if the paper is represented as a large tensor product state of quantum systems where each letter of the language is represented by a state of a component quantum system in the tensor product.

Additional details of a representation of language text in terms of the arrangements of ink molecules located on a 3 dimensional lattice of potential wells are described in the Appendix. This representation is just one of many possible. Others could be based on spin projection states of systems. More generally, one can use any physical observable with a discrete spectrum and eigenstates that can be associated with the language symbols. Also it must be possible to actually physically prepare systems in these eigenstates and to measure the properties of these systems corresponding to different properties of the language text.

This is similar to the $k-ary$ representation of numbers in quantum computation as tensor products of individual qukit states. In many physical models these tensor product states are represented by corresponding tensor product states of composite quantum systems. Each component state in the tensor product may also correspond to an entangled state of several quantum systems.\footnote{One can also construct representations of numbers in which there is no correspondence between the tensor product representation of qukits and the states representing numbers. An example of this using complex entangled states for $k=2$ was shown in \cite{BenEIPSRN} for numbers $ < 2^{n}$ with n arbitrary. A physical representation of numbers with entangled state structure representing that in the example is very unlikely to exist. The reason is that a necessary condition that states of quantum systems represent numbers is that the basic arithmetic operations be efficiently physically implementable \cite{BenRNQM,BenRNQMALG,BenEIPSRN}.} Examples of this are shown by various quantum error correcting codes \cite{QEC}.

The importance of this aspect is emphasized by the observation that, if it were not possible to represent language by states of physical systems, it would not be possible to communicate or acquire knowledge, or even think. It is an essential part of the existence of intelligent observers, as language is an essential part of the communication of information.

The basic requirement that language is physical is a different way to express Landauer's point that information is physical \cite{LandauerIP}. It has the advantage that for formal languages one can relate the various mathematical logical properties of these languages and their interpretations to their physical representations. It also clearly separates the concept of meaning of the language from the information content of the language. That is, the representation of language or information by states of physical systems is quite independent of whether the language or information has any meaning and, if so, what the meaning is.

One way to prove this independence is to note that for any axiomatizable theory there exists a computer program of finite length that can generate all the axioms of the theory. This is based on the decidability for any formula whether it is or is not an axiom of the theory. It follows from this that there exists a computer program of finite length that can enumerate all the theorems of the theory even though as is well known it is not decidable if a formula is or is not a theorem. Based on this the information content of an axiomatizable theory is defined to be the length of the shortest computer program that can enumerate the theorems of the theory \cite{Chaitin}.

Assume the theory is consistent. Then the formulas of the theory have meaning in that there exists a model of the theory in which the formulas have meaning and are true or false, based on their interpretation in the model. In particular the theorems are true.

Now create a new theory by adding a new axiom which is the negation of one of the original axioms. The information content (in the sense of the shortest program required to generate the theorems) of the new theory is only slightly increased over that of the original theory as the new axiom is a copy of one of the original axioms with a negation symbol added. However the extended set of axioms is inconsistent so the new theory has no meaning in that it has no model that satisfies the axioms. As a result all the formulas and theorems of the extended theory are meaningless.

It follows that two theories exist that have essentially the same information content but are radically different regarding their meaning. One has meaning and the other does not.

Another interesting aspect of a coherent theory of mathematics and physics that relates to the condition that language is physical, depends in part on the significance of G\"{o}del maps. These maps play an essential role in the proofs of the G\"{o}del incompleteness theorems for arithmetic and for any theory containing arithmetic \cite{Godel,Smullyan}.

The proof of G\"{o}del's first incompleteness theorem is based on the construction of an arithmetic sentence (a formula with no free variables) that can be interpreted to express its own unprovability. If the formula is false, then, since arithmetic is consistent, the formula cannot be a theorem as no false formula is a theorem in consistent theories. So it must be true. But then it is unprovable and is not a theorem. The negation of the formula is also not a theorem as it is false.

In arithmetic G\"{o}del maps are used to construct an arithmetic sentence referring to its own unprovability. In essence a G\"{o}del map is a map from the symbols, words, and expressions of the language of a theory to the elements of a model universe or domain of applicability of the theory. It enables any observer who knows the map to interpret formulas of the theory as referring to their metatheoretical properties. These are properties that are outside the domain of applicability of the theory. Typically they characterize the different types of expressions, such as whether or not expressions are variables, terms, formulas, sentences, axioms, proofs, etc.. But any metaproperty that can be described by a formula of the language by means of a G\"{o}del map is included.

For arithmetic there are many examples of different G\"{o}del maps in the literature. An especially transparent one was suggested by Quine \cite{Quine}. (See \cite{Smullyan} for a description.) To generate the map add one extra symbol to the alphabet to stand for a spacer. Then all expressions in the language consist of words and word sequences separated by one or more spacer symbols. If the extended alphabet has $n+1$ symbols, then the map assigns the numbers $0,1,\cdots ,n$ to the symbols.\footnote{These numbers are represented by $n+1$ numerals or digits.} The map is extended to all expressions by letting each expression correspond to a number in the base $n+1$, that uses the assignment of numbers to symbols.

A specific example in the literature uses 12 alphabet symbols and a spacer for arithmetic \cite{Smullyan}. Then each expression corresponds to a $13-ary$ representation of a number. For ordinary English, including the ten numerals and a spacer symbol but excluding punctuation marks, each expression as a string of words would correspond to a $37-ary$ representation of a number.

These maps can be used in arithmetic to show that there are arithmetic or number theoretic formulas that can be interpreted to express all the metatheoretic properties used in the definition of provability or theoremhood for a formula. These formulas can be used to construct an arithmetic formula, which is just a statement of some property of numbers, that can be interpreted to say that the nth formula in the G\"{o}del numbering is not a theorem. This formula is used along with some additional steps to construct an arithmetic formula that expresses its own unprovability, which is needed for the proof.

The reason for this somewhat detailed diversion into G\"{o}del maps is to emphasize the fact that for mathematical theories, such as arithmetic, the formulas are about numbers, or other suitable mathematical objects in the universe of discourse of the theory. Since the formulas themselves are not numbers or mathematical objects, none of their properties can be expressed by the formulas of the theory. It is the purpose of the G\"{o}del map to bridge this gap so that formulas can be interpreted as numbers. However the choice of the G\"{o}del map, limited only by some weak conditions that must be satisfied \cite{Smullyan}, is arbitrary and is up to the observer. It is also completely external to and is not a part of the theory to which it is being applied.

For theories such as a coherent theory of mathematics and physics, that is universally applicable (see Section \ref{UA} for a discussion), the situation is different and rather interesting. In essence the point is that any physical representation of a language with its symbols, words, and expressions is necessarily in the domain of applicability of a coherent theory of mathematics and physics, or of any theory that is applicable to all physical systems. Since a physical representation is an essential component of a language, it follows that a coherent theory must describe the physics of any systems whose states represent the language and formulas of the theory.

It is to be emphasized that the formulas of a coherent theory that is universally applicable, or of any physical theory, describe the physics of systems in their domain of applicability. They make no mention of which states of which physical systems in the domain represent symbols and formulas in the language of the theory.

However, each representation of symbols and formulas of the language by states of some physical systems corresponds to a G\"{o}del map. This includes the representation described in the Appendix. This is the case because any $1-1$ map from the symbols and formulas of the theory language to states of physical systems in the theory domain enables anyone who knows the map to interpret some formulas of the language as describing properties of the language and theory even though the formulas describe physical properties of physical systems. Since there are many representations possible, and each representation corresponds to a G\"{o}del map, there are many possible G\"{o}del maps. But one must know which map is being used to interpret specific states of physical systems as symbols and formulas in the language of the theory.

This mirrors the usual situation of the well known use of G\"{o}del maps in arithmetic. Formulas of the language of arithmetic describe properties of numbers, which are the mathematical systems in their domain of applicability. They make no mention of properties of formulas or formula sequences of the language. G\"{o}del maps enable anyone who knows the map to interpret arithmetic formulas that describe properties of numbers as describing properties of formulas and formula sequences and other properties of the language and theory of numbers. Since there are many G\"{o}del maps, there are many interpretations possible.

The situation discussed here is different from the well known use of classical or quantum computers to represent numbers. In this case a representation is a map from the natural numbers, integers, or rational numbers, as elements of model universes for the corresponding theories of these number types, to classical or quantum states of physical systems \cite{BenRNQM,BenRNQMALG,BenEIPSRN}. Functions correspond to operations represented as operators on the physical states of the systems. If the functions are one-one the corresponding operators are unitary for quantum system states.

The map does not correspond to a G\"{o}del map that maps all expressions of the language, including numeral strings and others interpreted as functions and relations, to physical states of physical systems. This latter situation is the focus of this work, but for theories encompassing both mathematics and physics.

The importance of these G\"{o}del maps for a coherent theory of mathematics and physics that is universally applicable should be noted. For mathematical theories, the condition that language is physical implies the existence of physical representations of the symbols, expressions, and sequences of expressions of the language. However these representations do not correspond to G\"{o}del maps because they are outside the domain of applicability of the theory. These representations are not maps from expressions in the language of the theory to elements of a model of the theory.

The situation is different for physical theories whose domains are specific types of physical systems or are part of the physical universe. In this case, depending on the physical systems described by the theory, there may be representations or G\"{o}del maps of the expressions of the theory language as states of physical systems that are described by the theory. However there are also in general other G\"{o}del maps of the language expressions to states of physical systems not in the domain of the theory.

An example of this would be a physical theory describing the dynamics of ink molecules as particles with many closely spaced internal states of excitation, moving on a two dimensional space lattice of potential wells. The dynamics would be described by a specific model Hamiltonian. As is seen in the Appendix, there exist G\"{o}del maps of formulas and sequences of formulas in the language of the theory to states of ink molecules on the lattice.

However there also exist other G\"{o}del maps of the formulas to states of other physical systems not described by the theory. An example would be maps of the language symbols onto spin projection eigenstates of systems also on a two dimensional space lattice where the systems and their dynamics are not described by the theory in question. In this case the physics of these systems would be described by another theory having different axioms than the theory of ink molecules.

This option is not available for a coherent theory that is universally applicable. In this case all G\"{o}del maps as physical representations of properties of the language of the theory to states of physical systems are in the domain of the theory. For any such map there must be formulas of the language expressing properties of physical systems that can be interpreted by anyone knowing the map as statements about properties of the formulas and of the theory itself.

This condition that all G\"{o}del maps or physical representations are necessarily in the theory domain might be expected to have some powerful consequences for a coherent theory of mathematics and physics that is universally applicable. This is especially the case if the theory satisfies the maximal validity and completeness requirement. That is it maximally describes its own validity and completeness and is maximally valid and complete. What these consequences are, if any, are not known at present.

\section{Discussion}\label{D}
At this point it is not known how to construct a coherent theory of mathematics and physics. However the material presented here may help in that it should be regarded as a general framework for constructing such a theory. The details and many aspects of a coherent theory remain to be worked out. Also some or many of the points and aspects described may need modification. However some

of the points are expected to remain.

First and foremost among the remaining points is the requirement that the coherent theory maximally describe its own validity and completeness and that it be maximally valid and complete. This requirement is expected to greatly restrict the range of allowed theories. It may even be so restrictive that just one theory satisfies it.

The requirement also has the advantage that it automatically ensures that any theory satisfying it agrees with experiment. This follows from the definition of validity, that any physical property that is predictable by the theory and is testable by experiment, is true. Maximal completeness ensures that the theory is maximally universally applicable as far as its predictive power is concerned. In other words the predictive power of the theory must be as strong or powerful as is possible.

This raises the problem that if the requirement that the theory agree with experiment is built into the structure of the theory itself, then one might think that the theory is not falsifiable or even testable. This is not the case. Even if the maximal validity and completeness requirement is built into the theory it still must be tested. In particular the theory may be interpreted to state, by complicated expressions, that it satisfies the requirement. This would include a statement of maximal agreement with experiment. But it this in fact the case? Is the theory statement of this true or false? One still has to carry out experiments to find out.

One should also keep separate the requirement that the theory maximally agree with experiment from what the actual results are of carrying out the experiments. For instance, incorporation of the maximal validity and completeness requirement into the theory may mean that the theory describe the existence of a map between a set of theoretical predictions and a set of experimental procedures, which are both described by the theory. The theory would also describe general properties of the map that correspond to agreement between theory and experiment.

However existence and general description of such a map in a coherent theory does not mean that a coherent theory is any different than present day physics regarding the need to carry out experiments to test the validity of theoretical predictions and determine detailed properties of the map. A coherent theory that satisfies the maximal validity and completeness requirement may deepen the understanding between physics and mathematics, and may even suggest new experiments, but it should not change the status or need to carry out experiments.

It also may be the case that the most basic aspects of the physical universe are a direct consequence of the basic requirement that there exist a coherent theory that maximally describes its own validity and completeness and is maximally valid and complete. Included are the reasons why space-time is $3+1$ dimensional, why quantum mechanics is the correct physical theory, and predictions of the existence and strengths of the four basic forces. However other aspects of the universe, which are also predictable by the theory, are not in this category and are subject to experimental test. This includes essentially all of the experimental and theoretical work done in physics.

There are also other possibilities to consider. For instance it may be the case that there is no single coherent theory. Instead there may be a nonterminating sequence of coherent theories, with each theory more inclusive than those preceding it. If this is the case then the $n+1st$ theory may include in its domain the requirement that the preceding n theories all maximally agree with experiment. But there may be other theoretical predictions in the $n+1st$ theory that are not present in the first n theories whose experimental status is outside the domain of the $n+1st$ theory.

This is one reason why the caveat of \emph{maximal} is present. Each theory in the sequence would describe its own validity and completeness to the maximum extent possible and would be valid and complete (this includes agreeing with experiment) to the maximum extent possible. But the amount of maximality, if such a concept is meaningful, would be different for the different theories.

The possibility of a nonterminating sequence of theories is similar to the situation in the proof of G\"{o}del's second incompleteness theorem \cite{Shoenfield,Smullyan}. There it is proved that, in any theory strong enough to express its own consistency, the formula expressing the consistency of the theory is not a theorem of the theory. Also the theory can be extended to a more comprehensive theory in which the consistency of the original theory can be proved. But then the formula expressing the consistency of the extended theory is not a theorem of the extended theory.

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\appendix

\section{Appendix}

Many physical representations are possible for the symbols and expressions of a language. Here a representation will be described that is based on the presence or absence of systems in small potential wells located on a two dimensional lattice of points on a solid state matrix. The description here will be quite simple and will be limited to the representation only. No dynamics corresponding to the generation of sequences of expressions, such as those that correspond to proofs, will be discussed.

A physical representation considered here is a model for text on printed pages in that the systems in the potential wells are ink molecules. Each symbol corresponds to a specific pattern of occupied wells surrounded by unoccupied wells. Expressions correspond to paths of symbols on the lattice. A solid state matrix with all potential wells unoccupied corresponds to a blank page. Locations of the wells on the page are given by X,Y coordinates x,y. Multiple pages can be considered by extending the lattice into three dimensions where $X-Y$ planes for different values of Z correspond to different pages.

Each potential well may or may not be occupied by ink molecules. Here an ink molecule is a complex system with many closely spaced internal states of excitation. The molecules are easily excited by absorption of ambient light of all visible wavelengths, and the excited states quickly decay by emitting cascades of infrared photons as heat or by transfer of phonons to the solid state matrix.

The state of an ink molecule in the ground state of the potential well at x,y in thermal equilibrium with an environment at temperature T is given by
\begin{equation}
\rho_{x,y} = \sum_{E}\frac{e^{-E/kT}}{Z}|E,0\rangle_{x,y}\langle 0,E|. \label{rhoxy} \end{equation} Here $|E,0\rangle_{x,y}$ denotes the ink molecule in a state with excitation energy E and in the ground state of the well located at lattice site x,y. Z is the partition function that normalizes the state. It is also assumed that the combination of the shape and height of each potential well and separation of the lattice points are such that the states $|E,0\rangle_{x,y}$ and $|E,0\rangle_{x^{\prime},y^{\prime}}$ are essentially orthogonal whenever $x\neq x^{\prime}$ or $y\neq y^{\prime}$.

To keep things simple the assumption is made that the energy spacing of the potential well states is large compared to kT where k is Boltzman's constant. Based on this Eq. \ref{rhoxy} is a good approximation to the state of the ink molecule in a well at x,y as the probability of being in a state above the ground state of the well is very small. It is also assumed that the internal excitation state of an ink molecule is essentially independent of whether the environment is visibly dark or well illuminated with visible light, provided only that both environments are at the same temperature.

The environmental bath also plays an important role in stabilizing the position states of the individual ink molecules to eigenstates of the individual potential wells. For example ink molecule states of the form $$\sum_{E}\frac{e^{-E/kT}}{Z}\sum_{x,y,x^{\prime},y^{\prime}} c_{x,y}c^{*}_{x^{\prime},y^{\prime}}|E,0\rangle_{x,y}\langle 0,E|_{x^{\prime},y^{\prime}}$$ would immediately decohere and stabilize \cite{Zurek,Zeh} to the diagonal form $\sum_{x,y}|c_{x,y}|^{2}\rho_{x,y}$ with $\rho_{x,y}$ given by Eq. \ref{rhoxy}.

Let α be an arbitrary finite set of points on a lattice. The quantum state corresponding to one ink molecule in each well at all locations in α and all other wells unoccupied is given by

\begin{align}\rho_{\alpha} & = \bigotimes_{x,y\epsilon \alpha}\rho_{x,y} \nonumber \\ \mbox{} & = \bigotimes_{x,y\epsilon \alpha}\sum_{E_{x,y}}\frac{e^{-E_{x,y}/kT}}{Z}|E_{x,y},0\rangle_{x,y}\langle 0,E_{x,y}|. \label{rhoal} \end{align}

Here symbols of a language correspond to sets of different patterns of closely spaced occupied wells. To this end let α_{S} be the set of occupied locations corresponding to the symbol S. A potentially useful characterization of the set α_{S} is in terms of a location x,y that serves as a standard fiducial mark or location parameter for the symbol, and a set b of scaling and other parameters needed to uniquely characterize the symbol S. Using this notation, which replaces α_{S} by $S_{x,y,b}$, Eq. \ref{rhoal} becomes \begin{equation} \rho_{S_{x,y,b}}=\bigotimes_{x,y\epsilon S_{x,y,b}}\rho_{x,y}. \label{rhoSxyb} \end{equation}

Some examples will serve to clarify this. The straight vertical line extending for n lattice sites in the Y direction from x,y to $x,y+n-1$ corresponds to the symbol $"|"$ located at x,y. The point x,y locating one end of the symbol serves as a fiducial location convention for this symbol. For each x,y the physical state of $"|"$ is given by $\rho_{|_{x,y,n}}= \otimes_{x^\prime =x}^{x+n-1}\rho_{x^{\prime},y}$ Other examples are the symbol $"/"$, a diagonal line of length n whose state is $\rho_{/_{x,y,n}}= \rho_{\alpha}$ with $\alpha =\{x,y;x+1,y+1;\cdots ;x+n-1,y+n-1\}$, and the $"\top "$ symbol with horizontal arm of length $2m+1$ and state description $\rho_{\top_{x,y,n,m}}= \rho_{\alpha}$ where $\alpha = \{x,y;\cdots ;x,y+n-1;x-m,y+n-1;\cdots ;x+m,y+n-1\}$. The values of n,m serve as scale factors for the symbols. For example if "$_{\top}$" is described by n,m, then "$\top$", which is the same symbol but is twice as large, would be described by $2n,2m$.

These examples and Eq. \ref{rhoSxyb} show the physical state representation for any printed symbol in a language for the printing model used here. However languages are composed of ordered sets or strings of symbols, as words. In many cases the language expressions may be further organized into strings of words. The printed text of this paper is an example of a long string of words.

For the representation considered here each word W corresponds to an unordered set α_{W} of occupied locations that corresponds to a set of disconnected sets α_{S} for each symbol S in W and an ordering of the symbol sets. The (unoccupied) spacing between the sets α_{S} should be larger that the spacing, if any between the individual ink molecule locations within each α_{S}. For reasons which will become clear soon, the ordering of the symbols S in W is separated from which symbols are in W. Note that the same symbol can appear more than once in a word.

Let B_{W} be the unordered set of symbols S in the word W. The state $\rho_{B_{W}}$ is given by $\rho_{B_{W}} =\otimes_{S\epsilon W}\rho_{\alpha_{S}}$ with $\rho_{\alpha_{S}}$

given by Eq. \ref{rhoal} with α_{S} replacing α. in
terms of fiducial marks and scale factors
\begin{equation}\rho_{B_{W\underline{x},\underline{y},\underline{b}}}
=\bigotimes_{S \epsilon B_{W}}\rho_{S,\underline{x}_{S}
,\underline{y}_{S},\underline{b}_{S}} \label{rhowd}\end{equation}
with $\rho_{S,\underline{x}_{S},
\underline{y}_{S},\underline{b}_{S}}$ given by Eq. \ref{rhoSxyb}.
The product is over all symbols S in the set B_{W} of
symbols in W at fiducial locations
$\underline{x}_{S},\underline{y}_{S}$ and scale factors
$\underline{b}_{S}$. Also $ \underline{x},\underline{y},
\underline{b}$ denote functions from the symbols in B_{W} to
x and y lattice positions and to a set of scale factors for
the symbols.

Since the lattice locations $\underline{x}_{S},
\underline{y}_{S}$ of the symbols in $\rho_{B_{W},\underline{x},
\underline{y},\underline{b}}$ are arbitrary, it is clear that
this state does not represent a word state. In order for this
state to represent a word W, the lattice locations of the
symbols must be ordered according to the rule by which the word
is read.

For many languages, texts are often organized into lines of
symbols in one space direction with successive lines ordered in
an orthogonal direction. Successive pages are then ordered in the
third space direction. Here spatial distances between symbols,
lines, and pages are used for the ordering. A specific example of
this has symbols in a line of text on a page with locations
corresponding to different values of x and a fixed value of
y. The symbols in the line are ordered according to their
positions in the X direction. Lines are ordered according to
their positions in the Y direction. Then if the values of
$\underline{x}_{S},\underline{y}_{S}$ in the definition of
$\rho_{B_{W}}$ are such that the $\underline{y}_{S}$ values are
all equal ($\underline{y}_{S}=y$ for all S) then the state
$\rho_{B_{W},\underline{x},\underline{y},\underline{b}}$ can be
identified as a state for a word W, or
$\rho_{B_{W},\underline{x},\underline{y},\underline{b}}=\rho_{W,\underline{x},
\underline{y},\underline{b}}$ where the letters in W are
ordered according to the ordering of their x positions.

For example let B_{W} consist of the letters $\{l,a,h,t\}$. If
the state $\rho_{B_{W}}$ is such that the y values for the
four symbols are equal and
$\underline{x}_{h}<\underline{x}_{a}<\underline{x}_{l}<\underline{x}_{t}$
then by Eq. \ref{rhowd}
$\rho_{B_{W},\underline{x},\underline{y},\underline{b}}$ is the
state for the word "halt" or
$\rho_{B_{W},\underline{x},\underline{y},\underline{b}}=\rho_{halt,\underline{x}
,
\underline{y},\underline{b}}$.

The point of this rather pedantic exercise is to emphasize that,
in this model, multisymbol states depend on the locations of the
component symbols and on the scaling or size of each symbol. In
addition the organization of these states into word states

depends on the rules used to read the words. These rules are given by the dynamics of the reading process. In addition the dynamics of this process are not completely arbitrary. They are subject to the requirement of efficient physical implementation. This requirement, which was discussed elsewhere in the context of representing numbers by states in quantum mechanics \cite{BenRNQM,BenRNQMALG,BenEIPSRN}, means that there must be a physical process which can read the text and that the space time and thermodynamic resources expended to implement the reading must be minimized. In particular the resources expended must not be exponential in the number of symbols read.

It is worth discussing this in a bit more detail. Consider for example symbols scattered about on an infinite $X-Y$ lattice in some state $\rho_{B_{W}}$. Any reading rule in which the determination of the location for reading the $n+1st$ symbol is based on what the first n symbols were requires an exponential amount of resources. This is based on the observation that if there are m symbols in the language, then for each n the rule must distinguish among m^{n} alternatives to make the determination.

Reading rules in use do not have this property in that determination of the location of the $n+1st$ symbol from the value of n does not depend on the state of the first n symbols. As such the rules are efficient in that the resources expended are polynomial in the number of symbols read. Since the requirement of polynomial efficiency is quite weak, there are many rules that satisfy this condition, so one would want to pick rules that more or less minimize the free energy resources expended. For example a rule in which the resources expended are independent of the number of symbols read (the polynomial exponent is 0) is quite efficient. The rule described for reading this text in English on the printed journal page is an example in that the resources expended to read the nth symbol are independent of n (except for carriage return locations). Other rules with the same or similar efficiency are possible, and some are in use.

The model described is clearly robust in the sense that reading the text does not change the individual symbol states or move them about on the lattice. Physically this is a consequence of the fact that photons in the visible light range excite the ink molecules to internal excited states. The potential wells and interactions with the component atoms in the molecules are such that the amplitudes for exciting an ink molecule to an excited well state, or to move it from one lattice location to another, are very small. It follows from this that sufficiently many repeated readings can move ink molecules around and make significant changes in the quantum state of the text.

Models that are much more sensitive to the reading interactions and other environmental influences can also be described. Examples would be spin models with spin systems replacing the ink molecules on the lattice. The projections of the spins along some direction such as that provided by a magnetic field would correspond to the different symbol states. Thus if the language contained m symbols, the spin $ \underline{S}$ of each of the systems must be such that $2\underline{S}+1\geq m$. The ordering of the symbol states of many systems into words and word sequences would depend on a path chosen on the lattice.

This model will not be pursued further here. Suffice it to say it is similar in some aspects to models of quantum NMR computing \cite{Gershenfeld,Favel,Chuang} which have been studied.

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